Early Fault-Tolerant Quantum Algorithms for Matrix Functions via Trotter Extrapolation

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- **Key Focus**: Develop algorithms with provable performance that are **resource-efficient** to be viable on existing quantum hardware.

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  - $f(A) = \Theta_x(A) \rightarrow$ find the ground state energy using Phase Estimation.

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All these techniques use $\mathrm{Tr}[Ze^{iHT}]$ which we estimate with our algorithm.

Background

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  - This leads to deep circuits, limiting applicability on near-term hardware.

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Method	Max Depth / Sample	Sample Overhead
Qubitization [4]	$\mathcal{O}\left(\Gamma\left[\Lambda T + \frac{\log(1/\varepsilon)}{\log\log(1/\varepsilon)}\right]\right)$	$\mathcal{O}(1/\varepsilon^2)$
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- **Commutator scaling:** Errors scale with nested commutators, which are often small in realistic systems. Performs substantially better when $\lambda_{\text{comm}} << \|H\|_1$

Our Algorithm and Results

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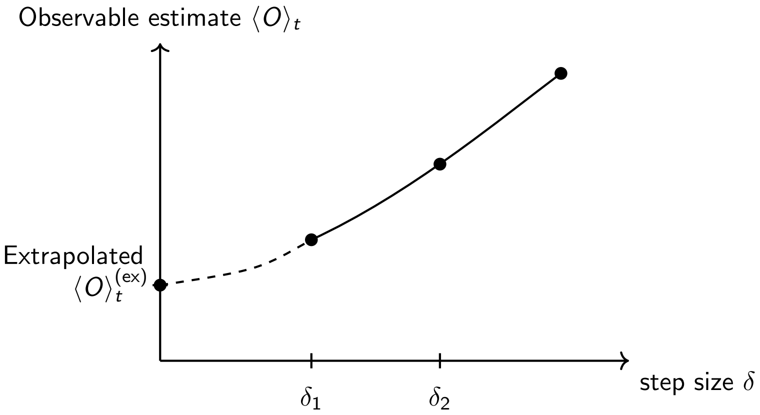
  - Perform quantum simulations at multiple Trotter step sizes and then **classically extrapolate** these results to for true evolution ($\delta \to 0$).



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- **Key Question**: Can this be applied to matrix functions?

Estimating General Matrix Functions

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We have it when $Z = \rho$, and extend to when $\|Z\|_1$ is bounded.

Richardson Extrapolation for each $\mathrm{Tr}[Ze^{iAt_k}]$

For each t_k, estimate $\mathrm{Tr}[Ze^{iAt_k}]$ via Richardson-extrapolated circuits.

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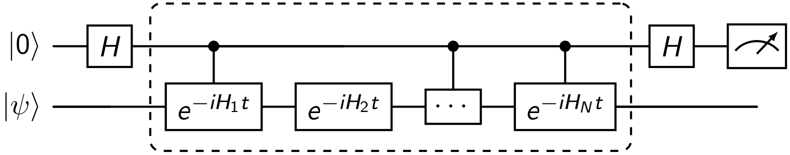
Each $\mathrm{Tr}[Z\mathcal{P}^{1/s_j}(s_j t_k)]$ is estimated by sampling from Hadamard circuits like the one below:



repeat $r_j := 1/s_j$ times

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Sample complexity $\mathcal{O}\left(\frac{\|Z\|_1^2 c^2 (\log\log(1/\varepsilon))^2}{\varepsilon^2} \cdot \log\left(\frac{1}{\delta}\right)\right)$

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allows to use and analyze Richardson extrapolation (to get coefficients $\{b_j\}_{j=1}^m$ and schedule $\{s_j\}_{j=1}^m$) for improved gate complexity.

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3. **Randomization**: improves sample overhead for circuits

Applications

Ground Energy Estimation via QPE ($f(A) = \Theta_x(A)$)

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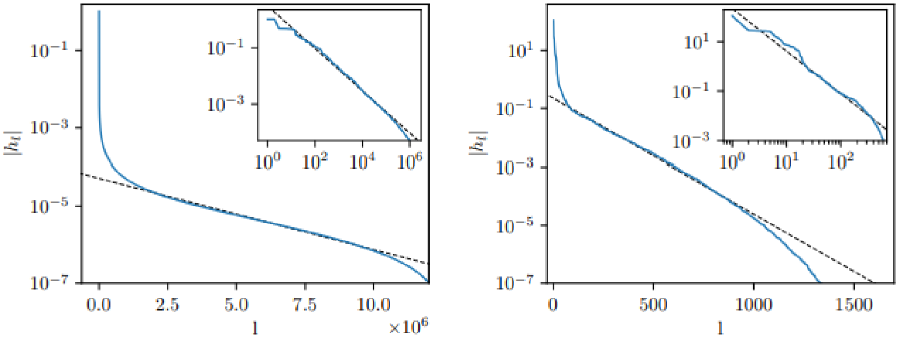
Refinements

Partial Randomization Motivation

Many Hamiltonians have relatively few high-weight terms. [1]



[1] Image from J. Günther, F. Witteveen, A. Schmidhuber, M. Miller, M. Christandl, and A. Harrow [7]

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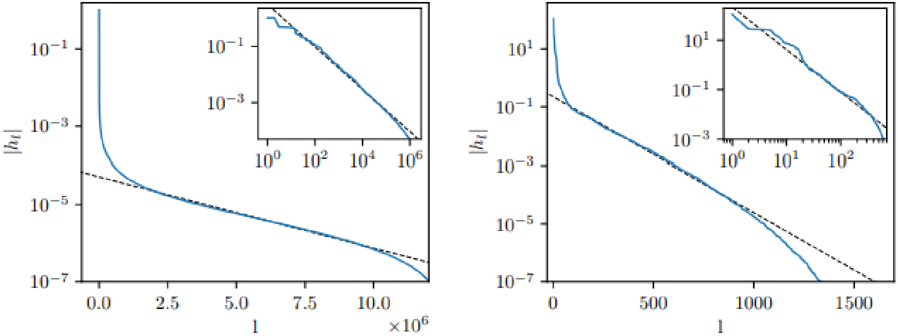

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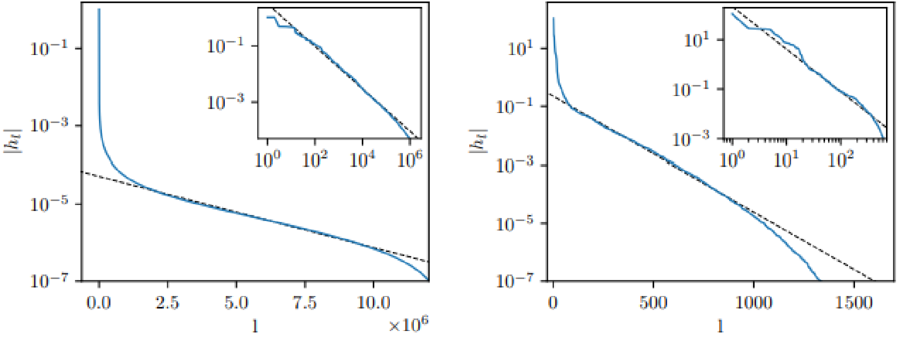

- The main plot shows exponential fit to the tail. ($|h_\ell| \approx Ae^{-b\ell}$)
- The insert shows power law fit for the large terms. ($|h_\ell| \approx C \cdot \ell^{-\alpha}$)

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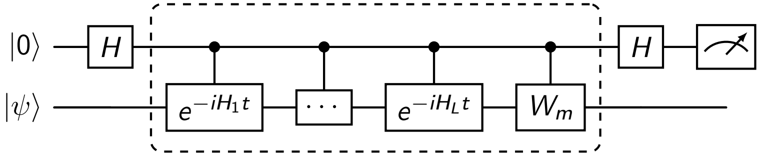
Richardson Extrapolation on Partial Randomization

We apply product formulas on L high weight terms and randomize the rest:



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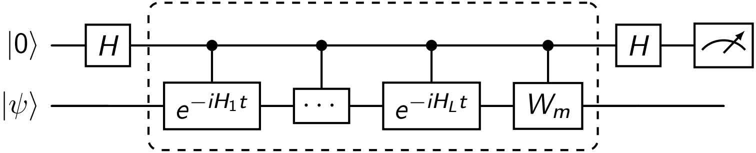

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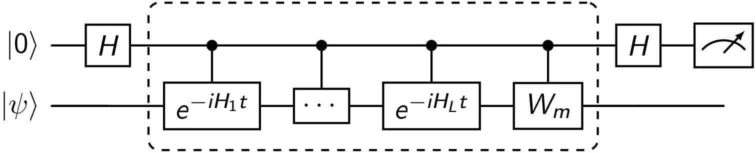

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To implement each W_m we use the Randomized Taylor Expansion [2].
We apply Richardson-extrapolation on the partially random circuits for:

$$C_{\text{gate}} = \mathcal{O}\left(L_D(\Upsilon\lambda_{\text{comm}}T)^{1+\frac{1}{p}}\log^2\left(\frac{1}{\varepsilon}\right) + \lambda_R^2 T^2\right)$$

$$C_{\text{sample}} = \mathcal{O}\left(\frac{1}{\varepsilon^2}\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^2\right)$$

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- Same bounds but with $\lambda_{\text{comm}}^{(\eta)}$ defined from $\alpha_{\text{comm}}^{(\eta)} < \alpha_{\text{comm}}$ [8]–[10] which are generally tighter than standard commutator bounds.

Next Steps:

- **1D extrapolation**: Note that the sample complexity

$$C_{\text{sample}} = \mathcal{O}\left(\frac{1}{\varepsilon^2}\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^4\right)$$

Can we improve the $\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^4$ to $\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^2$ by extrapolating directly over $\mathrm{Tr}[e^{iHT}\rho e^{-iHT'}O]$ instead of $\mathrm{Tr}[Ze^{iHT}]$?

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- **Analyzing constant factors** in the asymptotics for more specific performance estimates.

Acknowledgments

Samson Wang **John Preskill**

Thanks also to the rest of the Preskill Group, IQIM, and SFP for giving me such an enriching summer research experience.

Feedback:

PLEASE LEAVE YOUR AUDIENCE FEEDBACK HERE

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